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SHORTER COMMUNICATIONS

LAMINAR FLOW IN A POROUS PARABOLOIDAL PIPE

HSIAO C. KAO

Northrop Corporation, Aircraft Division, Hawthorne, California, U.S.A.

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NOMENCLATURE

C, D_1, D_3,	integration constants;
$f_1, f_2 \ldots$,	expanded terms of ϕ;
$R_s = -R_e\epsilon = \bar{\Psi}_0\epsilon/\bar{a}_0\nu$,	injection Reynolds number;
ϵ,	injection ratio;
η, ξ, q_N, q_T,	paraboloidal coordinates and dimensionless velocity components;
ν,	kinematic viscosity;
$\bar{\rho}$,	density;
$\phi, \bar{\Psi}$,	dimensionless and dimensional stream function respectively.

Superscript

bar,	dimensional quantity.

Subscript

0,	reference condition.

1. INTRODUCTION

PREVIOUS work on finding similarity solutions for the fully developed laminar flow in a porous pipe with injection has so far been limited to a circular pipe (see e.g. Berman [1] and Terrill and Thomas [2]). It is shown in this note that these solutions also exist for a porous paraboloidal pipe.

The analysis presented here is based on the asymptotic expansion of a small injection ratio, ϵ. In addition to this, there exists another dimensionless number, $R_e = \bar{\Psi}_0/\bar{a}_0\nu$ (the pipe Reynolds number). The multiplication of these two is, however, the controlling parameter. Thus, the double limit process is considered. There are at least two limiting cases, $R_e\epsilon = o(1)$ and $1/R_e\epsilon = O(1)$. These two bear some resemblance to those of supersonic and hypersonic flow past a slender body. In this connection, we mention parenthetically an intuitive notion. For hypersonic flows past axisymmetric slender bodies, we may think that the shock wave, which lies close to the body, forms an outer shell through which the air makes its entrance in a somewhat similar manner to that of liquid being forced into a permeable pipe.

2. DESCRIPTION OF THE PROBLEM

The problem to be considered is that of a paraboloidal pipe with porous walls through which the fluid is injected to the internal flow of same properties. The flow is assumed to be laminar, incompressible, and axi-symmetric. Under these conditions, the governing equations can be reduced to a single differential equation for the vorticity

$$\frac{\partial\bar{\Psi}}{\partial\bar{\eta}}\frac{\partial}{\partial\bar{\xi}}\left(\frac{\bar{\xi}\bar{\eta}}{\bar{\xi}^2+\bar{\eta}^2}\bar{D}^2\bar{\Psi}\right)-\frac{\partial\bar{\Psi}}{\partial\bar{\xi}}\frac{\partial}{\partial\bar{\eta}}\left(\frac{\bar{\xi}\bar{\eta}}{\bar{\xi}^2+\bar{\eta}^2}\bar{D}^2\bar{\Psi}\right)+2\left(\frac{\bar{\xi}}{\bar{\xi}^2+\bar{\eta}^2}\frac{\partial\bar{\Psi}}{\partial\bar{\xi}}-\frac{\bar{\eta}}{\bar{\xi}^2+\bar{\eta}^2}\frac{\partial\bar{\Psi}}{\partial\bar{\eta}}\right)\bar{D}^2\bar{\Psi}$$
$$=2\nu\bar{\xi}^2\bar{\eta}^2\bar{D}^2\left(\frac{\bar{\xi}\bar{\eta}}{\bar{\xi}^2+\bar{\eta}^2}\bar{D}^2\bar{\Psi}\right), \qquad (1)$$

where

$$\bar{D}^2=\frac{\partial}{\partial\bar{\xi}}\left(\frac{1}{\bar{\xi}\bar{\eta}}\frac{\partial}{\partial\bar{\xi}}\right)+\frac{\partial}{\partial\bar{\eta}}\left(\frac{1}{\bar{\xi}\bar{\eta}}\frac{\partial}{\partial\bar{\eta}}\right)$$

$$\bar{q}_T=\frac{1}{4\bar{\xi}\bar{\eta}(\bar{\xi}^2+\bar{\eta}^2)^{\frac{1}{2}}}\frac{\partial\bar{\Psi}}{\partial\bar{\eta}},\quad \bar{q}_N=\frac{-1}{4\bar{\xi}\bar{\eta}(\bar{\xi}^2+\bar{\eta}^2)^{\frac{1}{2}}}\frac{\partial\bar{\Psi}}{\partial\bar{\xi}}.$$

The symbols $\bar{\xi}$, $\bar{\eta}$ and others are the quantities in paraboloidal coordinates (see Fig. 1 and the Nomenclature).

To render this equation dimensionless, we define the following quantities

$$\phi=\frac{\bar{\Psi}}{\bar{\Psi}_0},\qquad \xi=\left(\frac{\epsilon}{\bar{a}_0}\right)^{\frac{1}{2}}\bar{\xi},\qquad \eta=(\bar{a}_0\epsilon)^{-\frac{1}{2}}\bar{\eta},$$
$$\bar{\Psi}_0=\bar{U}_0\bar{a}_0^2,\qquad \epsilon=\frac{-\bar{V}_0}{\bar{U}_0}. \qquad (2)$$

Here $\bar{\Psi}_0$ denotes the stream function at some station where the reference axial velocity component is $\bar{U}_0$, the reference injection velocity $-\bar{V}_0$, and the pipe radius $\bar{a}_0$. The symbol ϵ refers to the injection ratio.

The new variables ξ, η have now been stretched by the parameter ϵ which is assumed to be $o(1)$. The controlling parameter is, however, the combination of this and the pipe Reynolds number, $R_e\epsilon(R_e = \bar{\Psi}_0/\bar{a}_0\nu)$. For this there are two primary limiting cases. One is $R_e\epsilon = o(1)$ as $\epsilon \to 0$, and the other is $1/R_e\epsilon = O(1)$ as $\epsilon \to 0$. The first case has a formal analogue to the moderate supersonic flow past a slender body, where $M_\infty\tau \ll 1$ (M_∞ being the freestream Mach number and τ being the maximum body thickness ratio). The linearized method is applicable and the expansion is in powers of ϵ. Manton [3] has dealt with this problem in the context of a constricted pipe. Thus, the present study may be viewed as a complement to his work.

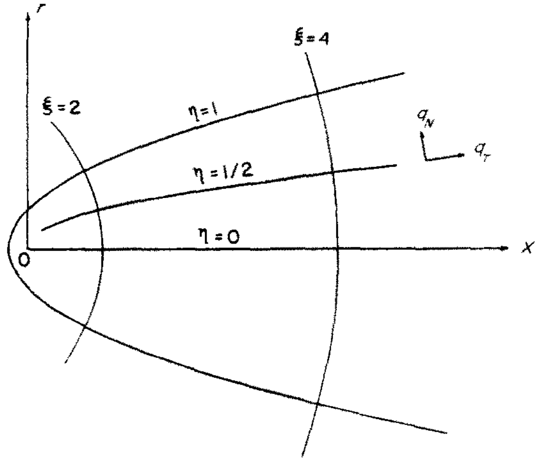

FIG. 1. Flow configuration.

The case of $1/R_e\epsilon = O(1)$ is analogous to the similarity parameter in the hypersonic small-disturbance theory (Hayes and Probstein [4]). Under this condition, the governing differential equation of the leading order is nonlinear. The asymptotic representation of ϕ is in powers of ϵ^2 instead of ϵ.

Returning to the formulation of the problem, we impose the following boundary conditions to equation (1) after rendering it dimensionless:

$$q_T(\xi, \eta_0) = 0 \qquad q_N(\xi, \eta_0) = \text{prescribed value} \tag{3}$$

$$\left(\frac{\partial q_T}{\partial \eta}\right)_{\eta=0} = 0, \qquad q_N(\xi, 0) = 0. \tag{4}$$

These boundary conditions in equation (3) and (4) are imposed along the pipe far downstream from the entrance region but as in the boundary layer theory these are not enough to define a unique solution (see e.g. Serrin [5]). To complement these conditions, we shall assign an initial velocity profile at some stations $\xi = \xi_0$ also far downstream from the entrance region where similar solutions subsist.

The end point $\eta = \eta_0$ at the porous surface in equation (3) can be replaced by $\eta = 1$ without loss of generality.

3. ANALYSIS OF THE PROBLEM

To establish similar solutions in a paraboloidal pipe, we assume that the injection ratio ϵ is small and the stream function ϕ can be expanded in an asymptotic series

$$\phi(\xi, \eta) = \xi^2 f_1(\eta) + \epsilon^2 f_2(\eta) + \epsilon^4 \xi^{-2} f_4(\eta) + \ldots \tag{5}$$

Notice that the condition of small ϵ can always be satisfied with an injection of constant flux throughout the permeable pipe, provided we proceed sufficiently downstream, where $-\bar{V}_0$ will be much less than $\bar{U}_0$.

Substituting equation (5) into the transformed equation (1) and equating terms of like magnitude of ϵ, we obtain the first-order approximation:

$$\eta^2 \frac{d}{d\eta}\left\{\frac{1}{\eta}\frac{d}{d\eta}\left[\eta \frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_1}{d\eta}\right)\right]\right\} - R_s\left\{f_1 \frac{d}{d\eta}\left[\eta\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_1}{d\eta}\right)\right] + \eta\frac{df_1}{d\eta}\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_1}{d\eta}\right) - 2f_1\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_1}{d\eta}\right)\right\} = 0 \tag{6}$$

and the second-order approximation.

$$\begin{aligned}\eta^2 \frac{d}{d\eta}\left\{\frac{1}{\eta}\frac{d}{d\eta}\left[\eta \frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_2}{d\eta}\right)\right]\right\} - R_s\left\{\eta\frac{df_1}{d\eta}\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_2}{d\eta}\right)\right. \\ + f_1\frac{d}{d\eta}\left[\eta\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_2}{d\eta}\right)\right] - \left(2f_1 - \eta\frac{df_1}{d\eta}\right)\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_2}{d\eta}\right) \\ \left. + \eta\frac{df_2}{d\eta}\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_1}{d\eta}\right)\right\} = \eta^2\frac{d}{d\eta}\left\{\frac{1}{\eta}\frac{d}{d\eta}\left[\eta^3\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_2}{d\eta}\right)\right]\right\} \\ - R_s\left\{\eta^3\frac{df_1}{d\eta}\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_1}{d\eta}\right) + f_1\frac{d}{d\eta}\left[\eta^3\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_1}{d\eta}\right)\right]\right. \\ \left. - \left(2\eta^2 f_1 - \eta^3\frac{df_1}{d\eta}\right)\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_1}{d\eta}\right)\right\}.\end{aligned} \tag{7}$$

To these differential equations we add the boundary conditions. In terms of the expansion coefficients, the velocity components q_T and q_N become

$$q_T = \frac{1}{4\eta}\frac{df_1}{d\eta} + \frac{\epsilon^2}{4\xi^2\eta}\left(\frac{df_2}{d\eta} - \frac{1}{2}\eta^2\frac{df_1}{d\eta}\right) + \ldots \tag{8}$$

$$q_N = \frac{1}{2\xi\eta}f_1 - \frac{\epsilon^2}{4\xi^3}\eta f_1 + \frac{\epsilon^4}{4\xi^5\eta}\left(\tfrac{3}{4}\eta^4 f_1 - 2f_4\right) + \ldots \tag{9}$$

To determine the prescribed value for q_N in equation (3), we assume that the flux through the porous surface per *unit axial distance*, x is constant. In this connection, we have

$$q_N = \frac{1}{2\xi\eta} - \frac{\epsilon^2}{4\xi^3}\eta + \frac{3\epsilon^4}{16\xi^5}\eta^3 - \ldots \tag{10}$$

From equations (3), (4) and (8)–(10) and the stipulation that the stream function ϕ being zero along the axis, the appropriate boundary conditions for f_1 and f_2 are

$$f_1(0) = 0, \lim_{\eta\to 0} \frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_1}{d\eta}\right) = 0; \qquad f_1(1) = 1, \left(\frac{df_1}{d\eta}\right)_{\eta=1} = 0 \quad (11)$$

$$f_2(0) = 0, \lim_{\eta\to 0} \frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_2}{d\eta}\right) = 0; \qquad \left(\frac{df_2}{d\eta}\right)_{\eta=1} = 0. \quad (12)$$

The first two stipulations in equation (11) imply that $q_N = 0$ as η tends to zero. It is seen in this process that one condition is still missing in equation (12). To complete this set, the initial condition at some station $\xi = \xi_0$ will be invoked.

Reverting now to the differential equations, it so happens that equation (6) can be integrated once and reduced to the following form:

$$\frac{d^3 f_1}{d\eta^3} - \frac{1}{\eta}\frac{d^2 f_1}{d\eta^2} + \frac{1}{\eta^2}\frac{df_1}{d\eta} - R_s \frac{f_1}{\eta}\left(\frac{d^2 f_1}{d\eta^2} - \frac{1}{\eta}\frac{df_1}{d\eta}\right) = C\eta \quad (13)$$

where C is the integration constant. Moreover, the function f_2 in equation (7) can also be expressed as quadratures of f_1:

$$f_2 = \int \left\{ \eta \exp\left(\int R_s \frac{f_1}{\eta} d\eta\right) \int \left[\eta^2 \frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df_1}{d\eta}\right) - R_s\left(f_1 \frac{df_1}{d\eta} - \frac{1}{\eta} f_1^2 - \frac{\eta}{2}\int \frac{1}{\eta}\left(\frac{df_1}{d\eta}\right)^2 d\eta\right) + \frac{D_1}{2}\eta \right] \exp\left(-\int R_s \frac{f_1}{\eta} d\eta\right) d\eta \right\} d\eta + D_3 \int \eta \exp\left(\int R_s \frac{f_1}{\eta} d\eta\right) d\eta, \quad (14)$$

where D_1 and D_3 are two surviving integration constants.

The pressure distribution in terms of the expanded variables is given by

$$P = -2C \ln \xi + \frac{\epsilon^2}{\xi^2}\left[-\frac{2}{\eta^2} f_1^2 + \int \frac{1}{\eta}\left(\frac{df_1}{d\eta}\right)^2 d\eta + \frac{D_1}{R_s}\right] + \ldots, \quad (15)$$

where P is defined as $P = \bar{P}/\bar{\rho}\bar{U}_0^2$ and C and D_1 are the same integration constants shown in equations (13) and (14). Although the injection Reynolds number, R_s, does not appear explicitly in the first-order pressure distribution, its influence is felt through the integration constant C.

4. CALCULATION AND DISCUSSION

The differential equation (13) along with the boundary conditions in equation (11) defines a two-point boundary-value problem, whose solution can be obtained by means of numerical method. The constant C is identified to be $C = 2/3(d^4 f_1/d\eta^4)_{\eta=0}$ and the region of integration is from $\eta = 0$ to $\eta = 1$.

Three sets of solutions at $R_s = -1{\cdot}0$, $-5{\cdot}0$ and $-10{\cdot}0$ have been computed. The final results are plotted in Fig. 2. (The curves for $R_s = -5{\cdot}0$ are not shown, since they are situated between the curves for $R_s = -1{\cdot}0$ and $-10{\cdot}0$.) Some pertinent values for initiating integration and for determining the wall shear stresses are shown in the following table.

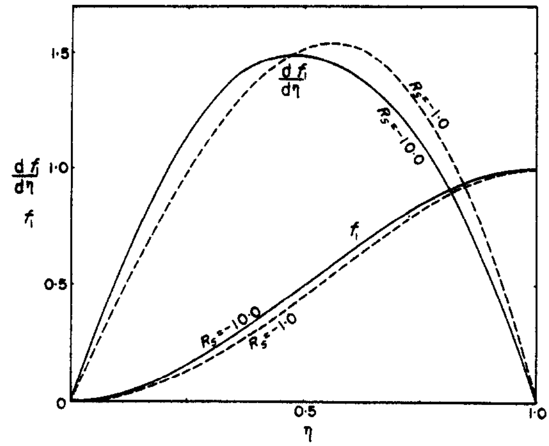

FIG. 2. Components of the first-order velocity profiles.

The quadratures in equation (14) are evaluated by applying Simpson's rule to the tabulated data of f_1 given at equal subintervals. The resulting profiles after choosing $f_2(1) = 0$ to be the supplementary condition are illustrated in Fig. 3.

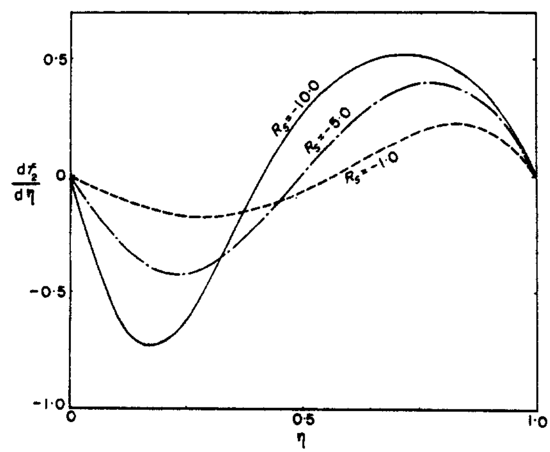

FIG. 3. Component $df_2/d\eta$ of the second-order velocity

R_s	$\left(\frac{d^2f_1}{d\eta^2}\right)_{\eta=0}$	$\left(\frac{d^4f_1}{d\eta^4}\right)_{\eta=0}$	$\left(\frac{d^2f_2}{d\eta^2}\right)_{\eta=0}$	$\left(\frac{d^4f_2}{d\eta^4}\right)_{\eta=0}$	$\left(\frac{d^2f_1}{d\eta^2}\right)_{\eta=1}$	$\left(\frac{d^2f_2}{d\eta^2}\right)_{\eta=1}$
−1·0	4·24	−30·5	−0·945	26·4	−7·41	−3·01
−5·0	4·90	−61·6	−2·97	152·0	−6·30	−3·91
−10·0	5·54	−109·0	−6·98	661·0	−6·23	−4·16

The dimensionless velocity gradient at the wall is given by

$$\left(\frac{\partial q_T}{\partial \eta}\right)_{\eta=1} = \frac{1}{4}\left(\frac{d^2f_1}{d\eta^2}\right)_{\eta=1} + \frac{\epsilon^2}{4\xi^2}\left(\frac{d^2f_2}{d\eta^2} - \frac{1}{2}\frac{d^2f_1}{d\eta^2}\right)_{\eta=1} + \ldots$$

The wall shear stress, which is proportional to the negative value of this quantity, is seen to be independent of ξ to the first approximation and decreases slightly from $R_s = -1{\cdot}0$ to $-10{\cdot}0$ (see the table). The effect of the second-order modification is generally small and diminishes with the increase of distance ξ. Thus, as in the hypersonic small-disturbance theory, the first-order approximation is expected to describe rather accurately the flow conditions in a porous paraboloidal pipe.

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THE HEAT BALANCE INTEGRAL METHOD*

DAVID LANGFORD†

Department of Mechanical Engineering, Drexel University, Philadelphia, Pennsylvania 19104, U.S.A.

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INTRODUCTION

APPROXIMATE solutions to transient diffusion problems may be obtained relatively easily by the use of what is commonly called "The Heat Balance Integral Method," THEBIM. THEBIM is applicable to one-dimensional linear and non-linear problems involving temperature dependent thermal properties [6, 7, 18], non-linear boundary conditions [7, 9], and phase change problems such as freezing [4, 5, 7–11, 17]. The applicability to phase change problems is of special importance [1, 2, 16] because existing closed form solutions to these significant problems are highly restrictive as to allowable initial conditions and boundary conditions [3, 12–15].

The accuracy of an approximate solution is in general unknown [2, 5–8, 11, 16]. Using THEBIM, attempts to increase the accuracy of an approximate solution have sometimes actually caused a decrease in accuracy [6, 7, 16]. There may therefore be some value in an accuracy criterion which can be easily used even when the exact solution is unknown. The use of such a criterion is illustrated here for a classical problem.

A SAMPLE PROBLEM

Let $T(x, t)$ be the temperature at position x at time t in a semi-infinite slab having constant thermal conductivity k,

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† Formerly Associate Professor of Mechanical Engineering at Drexel University, Phila. Pa. 19104. Now Regional Radiation Representative, U.S. Environmental Protection Agency, Phila., Pa. 19106, U.S.A.